

CLASSIFICATION OF CERTAIN CELLULAR CLASSES OF CHAIN COMPLEXES

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ABSTRACT. Let (R, \mathfrak{m}) be a local commutative ring. Suppose that \mathfrak{m} is principal and that $\mathfrak{m}^2 = 0$. We give a complete description of the cellular lattice of perfect chain complexes of modules over this ring.

1. INTRODUCTION

An explicit classification of thick subcategories of finite spectra ([HS98] and [DH95]) has been an important achievement of homotopy theory. This stable classification has been used to give an explicit classification of unstable Bousfield classes of finite suspension spaces [Bou96]. In contrast, an analogous classification of cellular classes of finite (suspension) spaces is out of reach as that would lead to a classification of ideals in the stable homotopy groups of spheres.

Thick subcategories of compact objects in the derived category of a ring are also well understood [Nee92]. Recently Bousfield classes (or Acyclic classes) of chain complexes have been classified, see [Sta] or [Kie]. However, as for spaces, the classification of cellular classes is more subtle. The aim of this paper is to give some examples of rings for which a classification can be obtained. For some properties of cellular classes of chain complexes over a Noetherian ring, see [Kie].

Throughout this paper all chain complexes are non-negatively graded chain complexes of modules over some fixed commutative ring.

Definition 1.1. Fix a chain complex A . We let $\mathcal{C}(A)$ denote the smallest collection of chain complexes satisfying the following properties: 1. The collection $\mathcal{C}(A)$ contains A , 2. It is closed under arbitrary sums and weak equivalences (i.e. homology isomorphisms) 3. If $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ is an exact sequence with $X, Y \in \mathcal{C}(A)$ then also $Z \in \mathcal{C}(A)$. If $X \in \mathcal{C}(A)$ then we write that $X \gg A$ and say that X is A -cellular.

In the paper we describe explicitly the cellular relations between perfect chain complexes of modules over a local ring (R, m) such that m is principal and $m^2 = 0$.

For such a ring and numbers $i, j \geq 0$, we let:

$$(\Sigma^i E_j)_n = \begin{cases} R & \text{for } i \leq n \leq i + j \\ 0 & \text{otherwise} \end{cases}$$

with the differentials given by multiplication by $(-1)^i r$, where r is some generator of the maximal ideal \mathfrak{m} . The isomorphism type of $\Sigma^i E_j$ does not depend on the choice of the generator r .

We prove that:

Theorem 1.2. *Let (R, m) be a local ring such that m is principal and $m^2 = 0$. Let A be a perfect chain complex that is not weakly equivalent to 0. Then there exists (i, j) such that:*

$$\mathcal{C}(A) = \mathcal{C}(\Sigma^i E_j)$$

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Moreover $\Sigma^{j'} E_{i'} \gg \Sigma^j E_i$ if and only if either $i' \geq i$ or $i' = i$ and $j' \geq j$.

The key element in the proof of the theorem is a classification of the perfect chain complexes. We show that any perfect chain complex A splits into a sum of a contractible chain complex and a sum of $\Sigma^i E_j$'s (see Lemma 5.2).

2. NOTATION

We let R denote some arbitrary commutative ring. By a *chain complex* X we mean a non-negatively graded chain complex of R -modules. We use the homological grading, i.e. the differential of X lowers the degree. Recall that the category of chain complexes of R -modules $Ch_{\geq 0}(R)$ is a model category [DS95]. In this model category a *weak equivalence* is a map that induces an isomorphism on homology. A *cofibration* is an injective map such that the cokernel is projective in each degree. A *fibration* is map which is surjective in all positive degrees. We let $\xrightarrow{\sim}$ denote a weak equivalence. A cofibrant chain complex X is a chain complex such that the canonical map $0 \rightarrow X$ is a cofibration, or explicitly, it is a chain complex of projective modules.

If $f : X \rightarrow Y$ is any map of chain complexes then we can factor $f = f''f'$, where $f' : X \rightarrow X'$ is a cofibration and $f'' : X' \xrightarrow{\sim} Y$ is a weak equivalence ([DS95]). Hence, any map is a cofibration up to a weak equivalence.

A complex X is called *perfect* if it is cofibrant and $\oplus_i X_i$ is finitely generated.

We let Hom denote the *hom-complex*. It is defined as follows: if $X, Y \in Ch_{\geq 0}(R)$ then $Hom(X, Y)_n = \prod_i hom(X_i, Y_{i+n})$ for $n > 0$ and $Hom(X, Y)_0$ is the set of maps of chain complexes from X to Y with the induced R -module structure. The differential takes $\{f_i : X_i \rightarrow Y_{i+n}\}_i \in Hom(X, Y)_n$ to $\{\partial f_i + (-1)^n f_{i-1} \partial\}_i$. If A is cofibrant, then $Hom(A, \bullet)$ preserves weak equivalences and fibrations (a consequence of Brown's lemma, see [DS95]).

We let Σ^i denote the *shift operator* i.e. $(\Sigma^i X)_j = X_{j-i}$ and $\partial_{\Sigma^i} = (-1)^i \partial$. The *cone* of a map $f : X \rightarrow Y$ is a chain complex $C(f)$ defined by: $C(f)_n = Y_n \oplus X_{n-1}$. The differential of $C(f)$ maps $(y, x) \in C(f)_n$ to $(\partial^Y(y) + f(x), -\partial^X(x))$. Note that there is a canonical map $Y \rightarrow C(f)$ and the cokernel of this map is isomorphic to $\Sigma^1 X$. If $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ is an exact sequence of chain complexes then there is a natural map from the cone of $X \rightarrow Y$ to Z . From the induced long exact sequences of homology, this map is a weak equivalence.

A more detailed account of the theory of chain complexes can be found for instance in [Wei94].

Associated to any module M are the *sphere complex*, $S^n(M)$, and the *disk complex*, $D^n(M)$, defined by:

$$(S^n(M))_i = \begin{cases} M & i = n \\ 0 & i \neq n \end{cases} \quad (D^n(M))_i = \begin{cases} M & i = n \text{ or } i = n-1 \\ 0 & \text{otherwise} \end{cases}$$

With the differential $\partial_n = 1_R$ in $D^n(M)$. For short we let $S^n := S^n(R)$ and $D^n := D^n(R)$.

3. CELLULAR RELATION

Recall that there is an alternative description of cellularity via a universal property (see [Far96]):

Proposition 3.1. *Let X and A be cofibrant chain complexes. Then X is A -cellular if and only if for all maps f such that $Hom(A, f)$ is a weak equivalence, the map $Hom(X, f)$ is also a weak equivalence.*

The cellular relation is transitive, in other words if $X \gg A$ and $Y \gg X$ then $Y \gg A$.

To determine whether a given chain complex X belongs to $\mathcal{C}(A)$ is in general a hard question. In Proposition 6.2 we give a workable criteria for detecting cellularity for a very particular choice of ring R . We now list certain properties of cellularity.

Proposition 3.2. (i) All chain complexes are S^0 -cellular.

(ii) If X is acyclic (i.e. all homology group vanish), then $X \gg A$ for any chain complex A .

(iii) The collection $\mathcal{C}(A)$ is closed under retracts.

(iv) Suppose that A is a cofibrant chain complex. If $X \gg A$ and $H_0(A) \neq 0$ then there is a set I and a map $f : \oplus_I A \rightarrow X$ such that $H_0(f)$ is surjective.

(v) If $X \gg A$ then $\Sigma^n X \gg A$ for all $n \geq 0$.

(vi) $\Sigma^1 X \gg \Sigma^1 A \Leftrightarrow X \gg A$.

Proof. We shall only give an outline of the proof. Statement (i) follows from the isomorphism $\text{Hom}(S^0, X) \cong X$ and 3.1. To prove the second statement note first that $0 \in \mathcal{C}(A)$ since $\text{Hom}(0, Y) = 0$. If X is acyclic then $0 \rightarrow X$ is a weak equivalence and since $\mathcal{C}(A)$ is closed under weak equivalences $X \gg A$. A retract of an isomorphism is an isomorphism so (iii) is a consequence of 3.1.

We fix a cofibrant chain complex A and let \mathcal{D} denote the collection of all chain complexes X such that there is a set I and a map $f : \oplus_I A \rightarrow X$ surjective on H_0 . It is a standard result in homological algebra ([Wei94], p. 388) that if $f : X \xrightarrow{\sim} Y$ is a weak equivalence and $g : A \rightarrow Y$ is any map, then because A is cofibrant, there is a map $h : A \rightarrow X$ such that g and fh are homotopic. As a consequence, the collection \mathcal{D} is closed under weak equivalences. It is also closed under sums. Finally if $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ is an exact sequence such that $X, Y \in \mathcal{D}$ then also $Z \in \mathcal{D}$. By definition $\mathcal{C}(A)$ was the smallest collection satisfying these properties, hence $\mathcal{C}(A) \subset \mathcal{D}$ and we have proved (iv).

Statements (v) and (vi) are direct consequences of 3.1. □

4. TWO OUT OF THREE PROPERTY

We say that a collection of chain complexes \mathcal{C} satisfies the *two out of three property* if given any exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ such that two out of X, Y and Z belong to \mathcal{C} then so does the third. The collection $\mathcal{C}(A)$ does not in general satisfy the two out of three property. For instance it follows from 3.2 that $\mathcal{C}(S^1)$ equals the collection of all chain complexes X such that $H_0 X = 0$. There is an exact sequence $0 \rightarrow S^0 \rightarrow D^1 \rightarrow S^1 \rightarrow 0$ and D^1 is S^1 -cellular, but S^0 is not.

Collections of possibly unbounded chain complexes of modules over a Noetherian ring, satisfying the two out of three property, that are closed under sums and weak equivalences have been classified by Neeman in [Nee92]. They are in 1-1 correspondence with arbitrary sets of prime ideals in R .

A collection \mathcal{C} of chain complexes is closed under extensions if given any exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ with $X, Z \in \mathcal{C}$ then also $Y \in \mathcal{C}$. The collection $\mathcal{C}(A)$ is in general not closed under extensions. In analogy with cellularity we can now define a relation called *acyclicity*:

Definition 4.1. Fix a chain complex A . Let $\mathcal{A}(A)$ denote the smallest collection of chain complexes satisfying the following properties: 1. The collection $\mathcal{A}(A)$ contains A , 2. It is closed under arbitrary sums, 3. If in an exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$, either X and Y or X and Z belong to $\mathcal{A}(A)$, then so does the third. If $X \in \mathcal{A}(A)$ then we write that $X > A$ and say that X is *A-acyclic*.

The collection $\mathcal{A}(A)$ is in particular closed under extensions. By definition $\mathcal{C}(A) \subset \mathcal{A}(A)$. This inclusion is in general strict:

Example 4.2. Recall from the introduction the definition of $E_i := \Sigma^0 E_i$. Up to a weak equivalence, E_1 is an extension of E_3 by E_2 : Let r denote some generator of the maximal ideal. Multiplication by r in degree 0 and the zero map in higher degrees defines a map $f : E_2 \rightarrow E_1$. The cone of f is isomorphic to E_3 . This gives the sequence: $E_2 \rightarrow E_1 \rightarrow E_3$

We later show (Theorem 6.3) that $E_3 \gg E_2$ (that is E_3 is E_2 -cellular) and that E_1 is *not* E_2 -cellular. However $E_1 > E_2$ since E_1 is an extension of E_3 by E_2 and $E_3 \gg E_2$.

The relation $>$ between perfect chain complexes of modules over a Noetherian ring is well understood. In fact $X > A$ if and only if for every n the following holds: if $p \subset R$ is a prime ideal such that $X_n \otimes R_p \neq 0$ then $A_i \otimes R_p \neq 0$ for some $i \leq n$. The proof of this result can be found in the papers [Sta] and [Kie].

The following Proposition establishes an important connection between cellularity and acyclicity. An analogous result for topological spaces was obtained by Dror-Farjoun in [Far96].

Proposition 4.3. *Fix a short exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$. If $X \gg A$ and $Z > \Sigma^1 A$ then $Y \gg A$.*

Proof. See [Kie]. A proof of the topological analog can be found in [Far96]. \square

5. A KEY LEMMA

For the rest of this paper we fix a local commutative ring R with a principal maximal ideal \mathfrak{m} such that $\mathfrak{m}^2 = 0$. We let k denote the residue field R/\mathfrak{m} . We choose some generator r of \mathfrak{m} . Note that in such a ring all non-unitary elements are of the form $x = r'r$, for some unit r' . Good examples to keep in mind are $R = \mathbb{Z}/(p^2)$ (p prime) and $R = k[X]/(X^2)$, for some field k .

Definition 5.1. An injective map $D^n \rightarrow X$ is called an *embedded disk*.

Recall from the introduction the special class of chain complexes:

$$(E_j)_n = \begin{cases} R & \text{if } n \leq j \\ 0 & \text{if } n > j \end{cases}$$

With $\partial_n : (E_j)_n \rightarrow (E_j)_{n-1}$ multiplication by r . For instance, E_∞ is a projective resolution of k and $E_1 = S^0$.

Lemma 5.2. *Let X be any perfect chain complex. Then there is a splitting of X :*

$$X \cong P \oplus Q$$

where P is acyclic and Q can be written as a finite sum

$$Q \cong \bigoplus \Sigma^i E_j$$

Proof. The proof is divided into several steps and takes up the rest of this section. Fix some perfect chain complex X .

Step 1

We first split off the contractible part of X .

Remark 5.3. R is injective as a module over itself. Hence D^n is an injective object, i.e. any embedded disk $D^n \rightarrow X$ is split.

A consequence of this remark is that X will split into a direct sum

$$X \cong P \oplus \tilde{X}$$

where P is an acyclic complex (a finite sum of disks) and \tilde{X} has no embedded disks.

Step 2

We can now assume that X has no embedded disks. We also want to assume that $X_0 \neq 0$.

Remark 5.4. For a perfect complex X , to have no embedded disks is equivalent to $\partial_n(X_n) \subset rX_{n-1}$ for all n . Such a complexes are also known as *minimal*.

Suppose that $H_0(X) = 0$. Then ∂_1 is surjective. From the remark we see that $X_0 = 0$. Hence $X = \Sigma^1 Y$ for some perfect complex Y .

Step 3

By step 1 and 2 we can assume that X is a perfect chain complex, containing no embedded disks and that $X_0 \neq 0$.

Since $H_0(X) \neq 0$ there is a surjection $X \rightarrow S^0(k)$ and because X is cofibrant this map factors:

$$X \rightarrow E_\infty \xrightarrow{\sim} S^0(k)$$

where $E_\infty \xrightarrow{\sim} S^0(k)$ is an acyclic fibration (i.e. a fibration and a weak equivalence).

The chain complex X is perfect, in particular $\oplus_i X_i$ is finitely presented, so the map $X \rightarrow E_\infty$ factors through some E_n . We note that the map $X \rightarrow E_n$ is surjective in degree 0. There is some smallest n_0 such that there is a map $f : X \rightarrow E_{n_0}$ surjective in degree 0. We claim that f is surjective and has a section, so that $X \cong E_{n_0} \oplus \tilde{X}$.

First we show that f is surjective. Suppose that this is not the case. Then there is some $m \leq n_0$ such that $\text{im } f_m \subset \ker \partial_m$. We truncate f at $m - 1$ and get a map $\tilde{f} : X \rightarrow E_{m-1}$ with $\tilde{f}_j = f_j$ for $j \leq m - 1$ and $\tilde{f}_j = 0$ for $j \geq m$. Then \tilde{f} is surjective in degree 0, contradicting the minimality of n_0 . Hence f is surjective.

We fix a generator e_j of $(E_{n_0})_j$ for each j . From the surjectivity of f_{n_0} it follows that there is some $x_{n_0} \in X_{n_0}$ such that $f_{n_0}(x) = e_{n_0}$. By remark 5.4 there is some x_{n_0-1} such that $rx_{n_0-1} = \partial(x_{n_0})$. Moreover $f_{n_0-1}(x_{n_0-1}) = a_{n_0-1}e_{n_0-1}$, for some unit a_{n_0-1} . Inductively we obtain a sequence of elements (x_0, \dots, x_{n_0}) and (a_0, \dots, a_{n_0}) . We define a map $s : E_{n_0} \rightarrow X$ by $s_j(e_j) = x_j$, $j \leq n_0$. This is well defined since $\partial \circ s_j(e_j) = rx_{j-1} = s_{j-1}(\partial(e_{j-1}))$. By construction $(f \circ s)_j(e_j) = a_j e_j$ so that $f \circ s$ is an isomorphism.

This determines a splitting of X into

$$X \cong E_{n_0} \oplus \tilde{X}$$

We can repeat the above discussion with \tilde{X} instead of X . This yields the required splitting formula. □

6. STATEMENT OF RESULTS

Recall that R is assumed to be a commutative local ring with a principal maximal ideal \mathfrak{m} such that $\mathfrak{m}^2 = 0$. We are now in a position to give a complete description of the cellular lattice of perfect complexes.

First we look at the cellularity relations among E_i 's. We begin with an observation.

Remark 6.1. If $i > j$ then there is no map $f : E_i \rightarrow E_j$, such that $H_0(f)$ is non-zero.

This remark in combination with the following proposition is enough to classify the cellular relations between the E_i 's.

Proposition 6.2. *Let X be a perfect complex such that $H_0(X) \neq 0$. Then Y is X -cellular if and only if there is a set I and a map $f : \oplus_{i \in I} X \rightarrow Y$ such that f induces an epimorphism on the H_0 .*

Proof. We first fix some perfect complex X with $H_0(X) \neq 0$. Let $D(X)$ denote the class of all complexes Y such that there is some I and $f : \oplus_I X \rightarrow Y$ with $H_0(f)$ surjective. The statement is that $D(X) = C(X)$.

By Proposition 3.2. To prove that $D(X) \subset C(X)$ we first note that since X is cofibrant, $X \otimes \bullet$ preserves cellularity: if $A \gg B$ then $A \otimes X \gg B \otimes X$. Here \otimes denotes the ordinary tensor product of chain complexes (see [Wei94]). Since $X \otimes S^0 \cong X$ and $S^0(k) \gg S^0$ we can conclude that $X \otimes S^0(k) \gg X$. We can always assume that there are no embedded disks in X and in this case $H_0(X \otimes k) \cong (X \otimes k)_0$. By assumption $H_0(X) \neq 0$, so there is a retraction: $S^0(k) \rightarrow X \otimes k \rightarrow S^0(k)$. It follows from Proposition 3.2 that $S^0(k) \gg X \otimes k \gg X$.

Fix a map $f : \oplus_{i \in I} X \rightarrow Y$ such that $H_0(f)$ is surjective. We can assume that $f : \oplus_{i \in I} X \rightarrow Y$ is a cofibration (see section 2). Let $Z := Y/f(\oplus_{i \in I} X)$. We wish to show that Y is X -cellular. By Proposition 4.3 it is enough to show that $Z > \Sigma^1 X$.

We assumed that $H_0(f_0)$ is surjective, so $H_0(Z) \cong 0$. As a consequence $Z \gg S^1$. There is an isomorphism of R -modules $\mathfrak{m} \cong k$. The exact sequence $0 \rightarrow k \rightarrow R \rightarrow k \rightarrow 0$ shows that $S^0 > S^0(k)$ or equivalently that $S^1 > S^1(k)$. In the paragraph above we showed that $S^1(k) \gg \Sigma^1 X$. In all we have that $Z \gg S^1 > S^1(k) \gg \Sigma^1 X$, i.e. $Z > \Sigma^1 X$. This concludes the proof of the proposition. \square

Consider the set S of all pairs (i, j) of non-negative integers. We order this set by declaring $(i, j) < (i', j')$ if either $i < i'$ or $i = i'$ and $j < j'$. To each perfect complex X we assign the subset S_X of S consisting of all pairs (i, j) such that $\Sigma^i E_j$ appears in the splitting of X as in lemma 5.2. Finally we let (i_X, j_X) denote the minimal element of S_X . We have shown that:

Theorem 6.3. *Let (R, m) be a local ring such that m is principal and $m^2 = 0$. Let X be a perfect complex over R . Then*

$$\Sigma^{i'} E_{j'} \gg \Sigma^i E_j$$

if and only if $(i, j) \leq (i', j')$. Moreover:

$$\mathcal{C}(X) = \mathcal{C}(\Sigma^{i_X} E_{j_X})$$

unless X is contractible, in which case $\mathcal{C}(X) = \mathcal{C}(0)$.

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